Existence, Asymptotic Behaviour, and Blow up of Solutions for a Class of Nonlinear Wave Equations with Dissipative and Dispersive Terms

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We consider the existence, both locally and globally in time, the asymptotic behaviour, and the blow up of solutions to the initial boundary value problem for a class of nonlinear wave equations with dissipative and dispersive terms. Under rather mild conditions on the nonlinear term and the initial data we prove that the above-mentioned problem admits a unique local solution, which can be continued to a global solution, and the solution decays exponentially to zero as $t \to +\infty$. Finally, under a suitable condition on the nonlinear term, we prove that the local solutions with negative and nonnegative initial energy blow up in finite time.

Key words: Nonlinear Wave Equation; Initial Boundary Value Problem; Global Solution; Asymptotic Behaviour; Blow up of Solutions.

1. Introduction

We are concerned with the existence, both locally and globally in time, the asymptotic behaviour, and the blow up of solutions to the initial boundary value problem for the following class of nonlinear wave equations with dissipative and dispersive terms:

$$u_{tt} - u_{xx} - u_{xxtt} - \lambda u_{xxt} + u = \sigma(u_x)_x, \quad (x,t) \in (0,1) \times (0,+\infty), \tag{1}$$

$$u(0,t) = u(1,t) = 0, \quad t \geq 0, \tag{2}$$

$$u(x,0) = u_0(x), \quad u_t(x_0) = u_1(x), \quad x \in [0,1], \tag{3}$$

where λ is a real number and $\sigma(s)$ is a given nonlinear function.

Physically, in real processes, the dissipation and dispersion have an important role for the energy amplification arising from the nonlinearity, and their interaction with the nonlinearity accompanies the accumulation, balance, and dissipation of the energy, see [1]. Many mathematicians and physicists focus their attention to study nonlinear evolution equations with dissipative or dispersive terms or with both of them. There are a lot of references investigating in detail the restriction conditions among the nonlinearity, the dispersion, and the dissipation, see [1 – 34].

The well known viscoelastic equation

$$u_{tt} - u_{xxt} = \sigma(u_x)_x \tag{4}$$

is an important class of nonlinear evolution equations which was suggested from the longitudinal displacement in a homogeneous rod with nonlinear strain and viscosity [5]. The dissipative term u_{xxt}, arising from the viscoelastic bar material, makes the initial boundary value problem of (4) more tractable than that of the one-dimensional nonlinear elasticity

$$u_{tt} = \sigma(u_x)_x.$$

There are many results [2, 6, 9, 10, 32] on the global existence, nonexistence and blow up, smoothness and asymptotic behaviours of solutions for the initial boundary value problem of (4).

In [32], Zhijian and Changming studied the blow up of solutions for the initial boundary value problem of (4).

In [35, 36], another class of nonlinear wave equations

$$u_{tt} - u_{xx} - u_{xxtt} = a\left(u_s^n\right)_x$$

was suggested in studying the transmission of nonlinear waves in a nonlinear elastic rod.

In [37], Guowang and Shubin proved the existence and uniqueness of a classical global solution and the blow up of solutions to the initial boundary value problem for the equation

$$u_{tt} - \alpha u_{xx} - \beta u_{xxtt} = \varphi(u_x)_x.$$

Finally they applied the results of the above problem to the equation arising for nonlinear waves in elastic rods:

$$u_{tt} - \left[a_0 + na_1(u_x)^{n-1}\right] u_{xx} - a_2 u_{xxtt} = 0.$$

Yacheng and Junsheng [31] studied the global existence, the asymptotic behaviour, and the blow up of $W^{k,p}$ solutions to the initial boundary value problem for the equation

$$u_{tt} - \alpha u_{xxt} - u_{xxtt} = \sigma(u_x)_x.$$

Zhijian [33] studied the global existence, asymptotic behaviour, and blow up of solutions to the initial boundary value problem for a class of nonlinear wave equations with a dissipative term:

$$u_{tt} + \Delta^2 u + \lambda u_t = \sum_{i=1}^{N} \frac{\partial}{\partial x_i} \sigma_i(u_{x_i}).$$

Polat et al. [23] established the blow up of solutions of the initial boundary value problem for a class of nonlinear wave equations with a damping term:

$$u_{tt} = \mathrm{div}\sigma(\nabla u) + \Delta u_t - \Delta^2 u.$$

Zhijian [34] has studied the existence, both locally and globally in time, the decay estimates, and the blow up of solutions to the Cauchy problem for a class of nonlinear dispersive wave equations arising in elastoplastic flow:

$$u_{tt} + u_{xxxx} + \lambda u = \sigma(u_x)_x,$$

and investigated the influence of the dispersive term λu for the corresponding solutions.

Levandosky [15] studied the local existence and decay estimates of solutions to the Cauchy problem of the equation

$$u_{tt} + \Delta^2 u + u = f(u).$$

Throughout the present paper, we use the following abbreviations and lemmas:

$$L^p = L^p[0,1] \quad \|\cdot\|_{L^p} = \|\cdot\|_p,$$

$$\|\cdot\|_{W^{k,p}[0,1]} = \|\cdot\|_{k,p}, \quad (u,v) = \int_0^1 uv\mathrm{d}x.$$

In order to simplify the exposition, different positive constants might be denoted by the same letter C.

Lemma 1 [31,33,38]. Let $\Omega \in \mathrm{R}^n$ be a bounded domain $k \geq 0$, $1 \leq p \leq \infty$. Assume that $G(z_1,\dots,z_h) \in C^k(\mathrm{R}^h)$, $z_i(x,t) \in L^\infty([0,T];W^{k,p}(\Omega))$ $(i = 1,\dots,h)$ and $\|z_i\|_{L^\infty([0,T];L^\infty(\Omega))}$. Then $G(z_1,\dots,z_h) \in L^\infty([0,T];W^{p,k}(\Omega))$ and

$$\|G(z_1,\dots,z_h)\|_{L^\infty([0,T];W^{p,k}(\Omega))} \leq C(M)\sum_{i=1}^{h} \|z_i\|_{L^\infty([0,T];W^{p,k}(\Omega))},$$

where

$$\|z\|_{L^\infty([0,T];W^{p,k}(\Omega))} = ess \sup_{0\leq t\leq T} \|z(t)\|_{k,p}.$$

Lemma 2 [31]. Assume that $f(s) \in C^{m+1}(\mathrm{R})$, $u,v \in L^\infty([0,T];W^{k,p}[0,1])$, $m \geq 1$, and $1 < p < \infty$. Then

$$\|f(u) - f(v)\|_{m,p} \leq C(\|u\|_{m,p}, \|v\|_{m,p})\|u - v\|_{m,p}, \quad 0 \leq t \leq T.$$

Lemma 3 [31,39,40]. Let $\Omega \in \mathrm{R}^n$ be a bounded domain, and $u(x) \in W^{1,2}(\Omega)$ be the unique solution of problem

$$\begin{aligned} &u - \Delta u = f(x), \quad x \in \Omega, \\ &u|_{\partial\Omega} = 0. \end{aligned} \tag{5}$$

Assume that $f(x) \in W^{k,p}(\Omega)$, $k \geq 0$, and $1 < p < \infty$, then $u(x) \in W^{k+2,p}(\Omega)$ and

$$\|u\|_{k+2,p} \leq C\|f\|_{k,p}.$$

Remark 1. Take $u = u(x,t)$ and $f = f(x,t)$ in Lemma 3, then the result of the lemma implies that $(I-\Delta)^{-1} : L^\infty([0,T];W^{k,p}(\Omega)) \to L^\infty([0,T);W^{k+2,p}(\Omega) \cap W_0^{1,p}(\Omega))$ and

$$\|(I-\Delta)^{-1}f\|_{L^\infty([0,T],W^{k+2,p}(\Omega))} \leq C\|f\|_{L^\infty([0,T],W^{k,p}(\Omega))},$$

where $(I-\Delta)^{-1}f = \int_\Omega K(x,y)f(y)\mathrm{d}y$, and $K(x,y)$ is the Green function of problem (5).

Lemma 4 [31,41–43]. Let $K(x,y)$ be the Green function of the boundary value problem for the ordinary differential equation

$$y(x) - y''(x) = 0, \quad y(0) = y(1) = 0, \tag{6}$$

i. e.,

$$K(x,\xi)=\frac{1}{\sinh 1}\begin{cases}\sinh(1-\xi)\sinh x, & 0\le x<\xi,\\ \sinh\xi\,\sinh(1-x), & \xi\le x\le 1.\end{cases}$$

The Green function $K(x,\xi)$ satisfies the following properties:

(1) $K(x,\xi)$ is defined and continuous in $Q=\{0\le x\le 1,\ 0\le\xi\le 1\}$.

(2) $K(x,\xi)$ satisfies the homogeneous equation

$$K(x,\xi)-K_{xx}(x,\xi)=0,\quad x\ne\xi$$

and the homogeneous conditions

$$K(0,\xi)=0,\quad K(1,\xi)=0.$$

(3) $K_x(x,\xi)$ has a point of discontinuity of the first kind at $x=\xi$ and satisfies the condition

$$K_x(\xi+0,\xi)-K_x(\xi-0,\xi)=-1.$$

(4) $K(x,\xi)=K(\xi,x)$.

(5) $0\le K(x,\xi)<\frac{2}{7},\quad 0\le x\le 1,\quad 0\le\xi\le 1.$

(6) $|K_{\xi,x}(x,\xi)|\le C,\quad x\ne\xi.$

The paper is organized as follows. First of all, we reduce problem (1) – (3) to an equivalent integral equation by means of the Green function of a boundary value problem for the second-order ordinary differential equation (6). Then making use of the contraction mapping principle we prove the existence and uniqueness of the local solutions for the integral equation in Section 2. Under some conditions by use of a priori estimates of the solution we prove in Section 3 that problem (1) – (3) has a unique global solution. The proof of the asymptotic behaviour of the global solutions is given in Section 4. In Section 5, the blow up of solutions for problem (1) – (3) is given.

2. Existence and Uniqueness of Local Solutions

In this section we prove the existence and the uniqueness of the local solutions for problem (1) – (3) by the contraction mapping principle.

For this purpose let $K(x,\xi)$ be the Green function of problem (6); we can rewrite (1) as follows:

$$[u_{tt}+u]-[u_{tt}+u]_{xx}=\sigma(u_x)_x+\lambda u_{xxt}. \tag{7}$$

From (7) and the solution of (1) satisfying condition (2), we get

$$\begin{aligned}&u_{tt}(x,t)+u(x,t)=\\&\left(I-\frac{\partial^2}{\partial x^2}\right)^{-1}(\sigma(u_x(x,t))_x+\lambda u_{xxt}(x,t))\\&\equiv\int_0^1 K(x,\xi)[\sigma(u_\xi(\xi,t))_\xi+\lambda u_{\xi\xi t}(\xi,t)]\mathrm{d}\xi.\end{aligned}\tag{8}$$

From (3) and (8) we know that the initial boundary value problem (1) – (3) is equivalent to the integral equation

$$\begin{aligned}u(x,t)&=u_0(x)+u_1(x)t-\int_0^t(t-\tau)u(x,\tau)\mathrm{d}\tau\\&\quad+\int_0^t(t-\tau)\left(I-\frac{\partial^2}{\partial x^2}\right)^{-1}(\sigma(u_x(x,\tau))_x\\&\quad+\lambda u_{xx\tau}(x,\tau))\mathrm{d}\tau\\&\equiv u_0(x)+u_1(x)t-\int_0^t(t-\tau)u(x,\tau)\mathrm{d}\tau\\&\quad+\int_0^t\int_0^1(t-\tau)K(x,\xi)[\sigma(u_\xi(\xi,\tau))_\xi\\&\quad+\lambda u_{\xi\xi\tau}(\xi,\tau)]\mathrm{d}\xi\mathrm{d}\tau.\end{aligned}\tag{9}$$

Now we are going to prove the existence and the uniqueness of the local solution for the integral equation (9) by the contraction mapping principle.

Let us define the function space

$$X_k(T)=\{u(x,t)\in W^{1,\infty}([0,T];W^{k,p}[0,1]\cap W_0^{1,p}[0,1]),\ u(0,t)=u(1,t)=0\},\tag{10}$$

which is endowed with the norm

$$\begin{aligned}\|u\|_{X_k(T)}&=\|u\|_{W^{1,\infty}([0,T];W^{k,p}[0,1])}\\&=\|u\|_{L^\infty([0,T];W^{k,p}[0,1])}+\|u_t\|_{L^\infty([0,T];W^{k,p}[0,1])},\\&\forall u\in X_k(T).\end{aligned}$$

It is easy to see that $X_k(T)$ is a Banach space. Let $M=\|u_0\|_{k,p}+\|u_1\|_{k,p}$. Take the set

$$Y_k(M,T)=\{u|u\in W^{1,\infty}([0,T];W^{k,p}[0,1]\cap W_0^{1,p}[0,1]),\ \|u\|_{X_k(T)}\le M+2\}.$$

Obviously, $Y_k(M,T)$ is a nonempty, bounded, closed convex subset of $X_k(T)$ for any fixed $M>0$ and $T>0$.

Define the map H as

$$Hu(x,t) = u_0(x) + u_1(x)t - \int_0^t (t-\tau)u(x,\tau)\mathrm{d}\tau + \int_0^t (t-\tau)\left(I - \frac{\partial^2}{\partial x^2}\right)^{-1} \cdot (\sigma(u_x(x,\tau))_x + \lambda u_{xx\tau}(x,\tau))\mathrm{d}\tau, \tag{11}$$

where $u \in X_k(T)$. We can easily show that H maps $X_k(T)$ into $X_k(T)$. If $\sigma(s) \in C^{k-1}(\mathrm{R})$, $k \geq 2$, $1 < p < \infty$, then from (10) and Lemma 1 we have

$$u_x \in W^{1,\infty}([0,T]; W^{k-1,p}[0,1] \cap L^\infty[0,1]),$$

$$\sigma(u_x) \in L^\infty([0,T]; W^{k-1,p}[0,1]),$$

and $\sigma(u_x)_x + \lambda u_{xxt} \in L^\infty([0,T]; W^{k-2}[0,1])$. By Lemma 3 we get

$$\left(I - \frac{\partial^2}{\partial x^2}\right)^{-1} (\sigma(u_x)_x + \lambda u_{xxt}) \in L^\infty([0,T]; W^{k,p}[0,1] \cap W_0^{1,p}[0,1]) \tag{12}$$

and $Hu \in X_k(T)$.

Our goal is to show that H has a unique fixed point in $Y_k(M,T)$ for suitable T.

Theorem 1. Assume that $u_0, u_1 \in W^{k,p}[0,1] \cap W_0^{1,p}[0,1]$ and $\sigma(s) \in C^k(\mathrm{R})$, $k \geq 2$, $1 < p < \infty$. Then H is a contractive mapping from $Y_k(M,T)$ into itself for T sufficiently small relative to M. Then problem (1)–(3) admits a unique solution $u(x,t) \in W^{2,\infty}([0,T_0); W^{k,p}[0,1] \cap W_0^{1,p}[0,1])$, where $[0,T_0)$ is the maximal time interval of existence for $u(x,t)$.

Proof. We first prove that H maps $Y_k(M,T)$ into itself for T small enough. Let $u \in Y_k(M,T)$ be given. From (11) we get

$$\|Hu\|_{k,p} \leq \|u_0\|_{k,p} + \|u_1\|_{k,p} T + \int_0^t (t-\tau)\|u\|_{k,p}\mathrm{d}\tau + \int_0^t (t-\tau)\left\|\left(I - \frac{\partial^2}{\partial x^2}\right)^{-1} (\sigma(u_x)_x + \lambda u_{xx\tau})\right\|_{k,p} \mathrm{d}\tau. \tag{13}$$

Using Lemma 3 and Lemma 1, it follows easily that

$$\|u\|_{k,p} \leq M+2, \quad \left\|\left(I - \frac{\partial^2}{\partial x^2}\right)^{-1} (\sigma(u_x)_x + \lambda u_{xxt})\right\|_{k,p} \leq C\|\sigma(u_x)_x + \lambda u_{xxt}\|_{k-2,p} \leq C(M)(M+2). \tag{14}$$

Substituting inequality (14) into (13) we obtain

$$\|Hu\|_{k,p} \leq \|u_0\|_{k,p} + \|u_1\|_{k,p} T + \frac{1}{2}(C(M)+1)(M+2)T^2. \tag{15}$$

On the other hand, from (11) and (14) we get

$$(Hu)_t = u_1 - \int_0^t u\mathrm{d}\tau + \int_0^t \left(I - \frac{\partial^2}{\partial x^2}\right)^{-1} (\sigma(u_x)_x + \lambda u_{xx\tau})\mathrm{d}\tau, \quad \|(Hu)_t\|_{k,p} \leq \|u_1\|_{k,p} + (C(M)+1)(M+2)T. \tag{16}$$

Thus from (15) and (16) we have

$$\|Hu\|_{X_k(T)} \leq M + (M + (C(M)+1)(M+2))T + \frac{1}{2}(C(M)+1)(M+2)T^2.$$

If T satisfies

$$t \leq \min\left\{\frac{1}{M + (C(M)+1)(M+2)}, \left[\frac{2}{(C(M)+1)(M+2)}\right]^{1/2}\right\}, \tag{17}$$

then

$$\|Hu\|_{X_k(T)} \leq M+2. \tag{18}$$

Therefore, if condition (17) holds, then H maps $Y_k(M,T)$ into $Y_k(M,T)$.

Let $T > 0$ and $u, v \in Y_k(M,T)$ be given. We have

$$\|Hu - Hv\|_{k,p} \leq \int_0^t (t-\tau)\|u-v\|_{k,p}\mathrm{d}\tau + \int_0^t (t-\tau)\left\|\left(I - \frac{\partial^2}{\partial x^2}\right)^{-1} [(\sigma(u_x) - \sigma(v_x))_x + \lambda(u-v)_{xx\tau}]\right\|_{k,p} \mathrm{d}\tau. \tag{19}$$

By Lemma 3 and Lemma 2, we obtain

$$\left\| \left(I - \frac{\partial^2}{\partial x^2} \right)^{-1} [(\sigma(u_x) - \sigma(v_x))_x + \lambda (u-v)_{xxt}] \right\|_{k,p} \leq C[C(M)\|u-v\|_{k,p} + \|(u-v)_t\|_{k,p}]. \quad (20)$$

Substituting inequality (20) into (19) we obtain

$$\|Hu - Hv\|_{k,p} \leq \frac{1}{2} C(M) T^2 \|u-v\|_{X_k(T)}. \quad (21)$$

On the other hand, from (11) and (20) we get

$$\|(Hu - Hv)_t\|_{k,p} \leq C(M) T \|u-v\|_{X_k(T)}. \quad (22)$$

Thus from (21) and (22) we have

$$\|Hu - Hv\|_{X_k(T)} \leq C(M) T \|u-v\|_{X_k(T)} + \frac{1}{2} C(M) T^2 \|u-v\|_{X_k(T)}.$$

Take T satisfying (17) and

$$T < \min \left\{ \frac{1}{2C(M)}, \left[\frac{1}{C(M)} \right]^{1/2} \right\}. \quad (23)$$

Then

$$\|Hu - Hv\|_{X_k(T)} < \|u-v\|_{X_k(T)}. \quad (24)$$

This shows that $H: Y_k(M,T) \to Y_k(M,T)$ is strictly contractive.

From (18) and (24) and the contraction mapping principle, for appropriately chosen $T > 0$, H has a unique fixed point $u(x,t) \in Y_k(M,T)$, which is a unique solution of problem (1) – (3). And from (8) and (12) we have

$$u_{tt}(x,t) + u(x,t) = \left(I - \frac{\partial^2}{\partial x^2} \right)^{-1} (\sigma(u_x)_x + \lambda u_{xxt}) \in L^\infty([0,T]; W^{k,p}[0,1] \cap W_0^{1,p}[0,1]). \quad (25)$$

Thus (25) implies

$$u(x,t) \in W^{2,\infty}([0,T_0); W^{k,p}[0,1] \cap W_0^{1,p}[0,1]), \quad (26)$$

where $[0,T_0)$ is the maximal time interval of existence for $u \in X_k(T_0)$. This completes the proof of the theorem.

Now, we discuss the global existence and uniqueness of solutions.

3. Existence and Uniqueness of Global Solutions

In this section, we prove the existence and uniqueness of the global solutions for problem (1) – (3). For this purpose we are going to make a priori estimates of the local solutions for problem (1) – (3) and we suppose that the conditions of Theorem 1 hold.

Theorem 2. Assume that $u_0, u_1 \in W^{2,p}[0,1] \cap W_0^{1,p}[0,1]$, $\sigma(s) \in C^2(\mathrm{R})$, $1 < p < \infty$, and the following conditions hold:

(i) $\sigma(s)s \geq 0$, $s \in \mathrm{R}$;

(ii) $|\sigma(s)| \leq C_1 \int_0^s \sigma(y) \mathrm{d}y + C_2$, $s \in \mathrm{R}$, where C_1 and C_2 are positive constants.

Then for any $T > 0$ problem (1) – (3) admits a unique global solution $u(x,t) \in W^{2,\infty}([0,T]; W^{2,p}[0,1] \cap W_0^{1,p}[0,1])$.

Proof. Taking the L^2 inner product with u_t in (1) and integrating the resulting expression over $[0,t]$ we get

$$\|u(t)\|_{H^1}^2 + \|u_t(t)\|_{H^1}^2 + 2\int_0^1 F(u_x)\mathrm{d}x + 2\lambda \int_0^t \|u_{x\tau}(\tau)\|_2^2 \mathrm{d}\tau = \|u_0\|_{H^1}^2 + \|u_1\|_{H^1}^2 + 2\int_0^1 F(u_{0x})\mathrm{d}x, \quad (27)$$

where $F(s) = \int_0^s \sigma(y)\mathrm{d}y$.

If $\sigma(s)s \geq 0$, $s \in \mathrm{R}$, then $F(s) \geq 0$. Thus from (27) we have

$$\|u(t)\|_{H^1}^2 + \|u_t(t)\|_{H^1}^2 + 2\int_0^1 F(u_x)\mathrm{d}x \leq \|u_0\|_{H^1}^2 + \|u_1\|_{H^1}^2 + 2\int_0^1 F(u_{0x})\mathrm{d}x + 2|\lambda| \int_0^t \|u_{x\tau}(\tau)\|_2^2 \mathrm{d}\tau.$$

From the above inequality and the Gronwall inequality we get

$$\|u(t)\|_{H^1}^2 + \|u_t(t)\|_{H^1}^2 + 2\int_0^1 F(u_x)\mathrm{d}x \leq C(T) \quad (28)$$

and

$$\|u(t)\|_{H^1}^2 + \|u_t(t)\|_{H^1}^2 \leq C(T), \quad 0 \leq t \leq T, \quad (29)$$

where $C(t)$ is a constant dependent on T.

With partial integration of (28), we obtain

$$\begin{aligned} u_{tt}(x,t)+u(x,t) &= \int_0^1 K(x,\xi)[\sigma(u_\xi(\xi,t))_\xi + \lambda u_{\xi\xi t}(\xi,t)]\mathrm{d}\xi \\ &= -\int_0^1 K_x(x,\xi)[\sigma(u_\xi(\xi,t)) + \lambda u_{\xi t}(\xi,t)]\mathrm{d}\xi. \end{aligned}$$

Differentiating the above equation with respect to x and using (3) of Lemma 4 it follows that

$$\begin{aligned} &u_{ttx}(x,t)+u_x(x,t) \\ &= -\lim_{\delta\to 0}\Big(\int_0^{x-\delta} + \int_{x-\delta}^{x+\delta} + \int_{x+\delta}^1 K_{\xi x}(x,\xi)[\sigma(u_\xi(\xi,t)) \\ &\qquad + \lambda u_{\xi t}(\xi,t)]\mathrm{d}\xi\Big) \\ &= -\lim_{\delta\to 0}\Big(\int_0^{x-\delta} + \int_{x+\delta}^1 K_{\xi x}(x,\xi)[\sigma(u_\xi(\xi,t)) \\ &\qquad + \lambda u_{\xi t}(\xi,t)]\mathrm{d}\xi\Big) \\ &\quad - \sigma(u_x(x,t)) - \lambda u_{xt}(x,t). \end{aligned} \tag{30}$$

Multiplying both sides of (30) by u_{xt} and using (6) of Lemma 4 we get

$$\begin{aligned} &\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}(u_{xt}^2 + u_x^2 + 2F(u_x)) + \lambda u_{xt}^2 \\ &\le C\int_0^1 (|\sigma(u_\xi)| + |\lambda u_{\xi t}|)\mathrm{d}\xi\, |u_{xt}|. \end{aligned} \tag{31}$$

By condition (ii) of Theorem 2, the Young inequality, and inequality (28) we have

$$\begin{aligned} &\int_0^1 (|\sigma(u_\xi)| + |\lambda u_{\xi t}|)\mathrm{d}\xi \le \\ &\int_0^1 \left(C_1F(u_\xi) + C_2 + \frac{1}{2}(u_{\xi t}^2 + \lambda^2)\right)\mathrm{d}\xi \le C(T). \end{aligned} \tag{32}$$

Substituting inequality (32) into inequality (31) and using the Young inequality we obtain

$$\begin{aligned} &\frac{\mathrm{d}}{\mathrm{d}t}(u_{xt}^2 + u_x^2 + 2F(u_x)) \\ &\le C_3(T) + \left(2|\lambda| + \frac{1}{2}\right)u_{xt}^2. \end{aligned} \tag{33}$$

Here $C_3(T)$ is a constant dependent on T. Integrating (33) with respect to t and using the Gronwall inequality we get

$$\begin{aligned} &\|u_{xt}\|_\infty^2 + \|u_x\|_\infty^2 + 2\|F(u_x)\|_\infty \le \\ &(\|u_1\|_\infty^2 + \|u_{0x}\|_\infty^2 + 2\|F(u_{0x})\|_\infty + C_3(T)T)\mathrm{e}^{(2|\lambda|+\frac{1}{2})T}. \end{aligned}$$

By assumption (i) of Theorem 2 and the above inequality we obtain

$$\|u_{xt}\|_\infty^2 + \|u_x\|_\infty^2 \le C(T), \quad 0 \le t \le T. \tag{34}$$

Using Lemma 3 and Lemma 1, it follows easily that

$$\begin{aligned} &\left\|\left(I - \frac{\partial^2}{\partial x^2}\right)^{-1}(\sigma(u_x)_x + \lambda u_{xxt})\right\|_{2,p} \\ &\le C(\|u\|_{2,p} + \|u_t\|_{2,p}). \end{aligned} \tag{35}$$

From (9) and (35) we obtain

$$\begin{aligned} \|u\|_{2,p} \le{}& \|u_0\|_{2,p} + \|u_1\|_{2,p}T \\ &+ T\int_0^t [(C+1)\|u\|_{2,p} + C\|u_\tau\|_{2,p}]\mathrm{d}\tau, \end{aligned} \tag{36}$$

$$\begin{aligned} \|u_t\|_{2,p} \le{}& \|u_1\|_{2,p} \\ &+ \int_0^t [(C+1)\|u\|_{2,p} + C\|u_\tau\|_{2,p}]\mathrm{d}\tau. \end{aligned} \tag{37}$$

Thus from (36) and (37) we have

$$\begin{aligned} \|u\|_{2,p} + \|u_t\|_{2,p} \le{}& \|u_0\|_{2,p} + \|u_1\|_{2,p}(1+T) \\ &+ C(1+T)\int_0^t (\|u\|_{2,p} + \|u_\tau\|_{2,p})\mathrm{d}\tau. \end{aligned}$$

Applying the Gronwall inequality to the above inequality we obtain

$$\|u\|_{2,p} + \|u_t\|_{2,p} \le C(T), \quad 0 \le t \le T. \tag{38}$$

From (25), (35), and (38) we have

$$\|u_{tt}\|_{2,p} + \|u\|_{2,p} \le C(\|u\|_{2,p} + \|u_t\|_{2,p}),$$

$$\|u_{tt}\|_{2,p} \le C(T), \quad 0 \le t \le T,$$

and

$$u(x,t) \in W^{2,\infty}([0,T]; W^{2,p}[0,1] \cap W_0^{1,p}[0,1]).$$

By the arbitrariness of T and (26), $T_0 = \infty$, Theorem 2 is proved.

Theorem 3. If $\sigma'(s)$ is bounded from below, i. e., there is a constant C_0 such that $\sigma'(s) \ge C_0$ and $|\tilde{\sigma}(s)| \le C_4\int_0^s \tilde{\sigma}(y)\mathrm{d}y + C_5$, $s \in \mathrm{R}$ where $\tilde{\sigma}(s) = \sigma(s) - k_0 s - \sigma(0)$, $k_0 = \min\{C_0, 0\} \le 0$, C_3 and C_4 are positive constants, then the conclusion of Theorem 2 also holds.

Proof. Let $\tilde{\sigma}(s) = \sigma(s) - k_0 s - \sigma(0)$, where $k_0 = \min\{C_0, 0\} \le 0$. Obviously $\tilde{\sigma}(0) = 0$, $\tilde{\sigma}'(s) = \sigma'(s) -$

$k_0 \geq 0$, and $\tilde{\sigma}(s)$ is a monotonically increasing function. Then $\tilde{F}(s) = \int_0^s \tilde{\sigma}(y)\mathrm{d}y \geq 0$. From (27) and noting that

$$F(s) = \int_0^s \sigma(y)\mathrm{d}y = \int_0^s [\tilde{\sigma}(y) + k_0 y + \sigma(0)]\mathrm{d}y,$$

we have

$$\begin{aligned}
&\|u(t)\|_{H^1}^2 + \|u_t(t)\|_{H^1}^2 + 2\int_0^1 \tilde{F}(u_x)\mathrm{d}x \\
&\leq \|u_0\|_{H^1}^2 + \|u_1\|_{H^1}^2 + 2\int_0^1 \tilde{F}(u_{0x})\mathrm{d}x \\
&\quad + k_0\|u_{0x}\|_2^2 + 2|\lambda| \int_0^t \|u_{x\tau}(\tau)\|_2^2\mathrm{d}\tau - k_0\|u_x(t)\|_2^2 \\
&= \|u_0\|_{H^1}^2 + \|u_1\|_{H^1}^2 + 2\int_0^1 \tilde{F}(u_{0x})\mathrm{d}x + k_0\|u_{0x}\|_2^2 \\
&\quad + 2|\lambda| \int_0^t \|u_{x\tau}(\tau)\|_2^2\mathrm{d}\tau - k_0\|u_0\|_2^2 - 2k_0\int_0^t (u, u_\tau)\mathrm{d}\tau \\
&\leq \|u_1\|_2^2 + (1-k_0)\|u_0\|_2^2 + (1+k_0)\|u_{0x}\|_2^2 + \|u_{1x}\|_2^2 \\
&\quad + 2\int_0^1 \tilde{F}(u_{0x})\mathrm{d}x \\
&\quad + \int_0^t [(2|\lambda|+1)\|u_\tau(\tau)\|_2^2 + k_0^2\|u(\tau)\|_2^2]\mathrm{d}\tau.
\end{aligned}$$

From the above inequality and the Gronwall inequality we get

$$\|u(t)\|_{H^1}^2 + \|u_t(t)\|_{H^1}^2 + 2\int_0^1 \tilde{F}(u_x)\mathrm{d}x \leq C(T), \quad 0 \leq t \leq T.$$

Therefore substituting $\sigma(s) = \tilde{\sigma}(s) + k_0 s + \sigma(0)$ into (1) and the other, and repeating the proof in Theorem 2 leads to the conclusions of Theorem 3.

Theorem 4. Assume that $u_0, u_1 \in W^{k,p}[0,1] \cap W_0^{1,p}[0,1]$, $\sigma(s) \in C^k(\mathrm{R})$, $k > 2$, $1 < p < \infty$, and the following conditions hold:

(i) $\sigma(s)s \geq 0, \quad s \in \mathrm{R}$;

(ii) $|\sigma(s)| \leq C_1 \int_0^s \sigma(y)\mathrm{d}y + C_2, \quad s \in \mathrm{R}$.

Then for any $T > 0$ problem (1)–(3) admits a unique global solution $u(x,t) \in W^{2,\infty}([0,T]; W^{k,p}[0,1] \cap W_0^{1,p}[0,1])$.

Proof. From Theorem 1 we know that problem (1)–(3) admits a unique local solution $u(x,t) \in W^{2,\infty}([0,T_0); W^{k,p}[0,1]$. From the proof of the theorem we have $u(x,t) \in W^{2,\infty}([0,T]; W^{2,p}[0,1] \cap W_0^{1,p}[0,1])$, $\forall T > 0$.

From (9) and (14), we have

$$\|u\|_{k,p} \leq \|u_0\|_{k,p} + \|u_1\|_{k,p}T + T\int_0^t [(C+1)\|u\|_{k,p} + C\|u_\tau\|_{k,p}]\mathrm{d}\tau, \tag{39}$$

$$\|u_t\|_{k,p} \leq \|u_1\|_{k,p} + \int_0^t [(C+1)\|u\|_{k,p} + C\|u_\tau\|_{k,p}]\mathrm{d}\tau. \tag{40}$$

Adding inequalities (39) and (40), and applying the Gronwall inequality to the resulting inequality, we obtain

$$\|u\|_{k,p} + \|u_t\|_{k,p} \leq C(T), \quad 0 \leq t \leq T. \tag{41}$$

From (25), (14), and (41) we have

$$\|u_{tt}\|_{k,p} + \|u\|_{k,p} \leq C(\|u\|_{k,p} + \|u_t\|_{k,p}),$$

$$\|u_{tt}\|_{k,p} \leq C(T), \quad 0 \leq t \leq T,$$

and

$$u(x,t) \in W^{2,\infty}([0,T]; W^{k,p}[0,1] \cap W_0^{1,p}[0,1]).$$

By the arbitrariness of T and (26), $T_0 = \infty$, Theorem 4 is proved.

4. Asymptotic Behaviour of Solutions

In this section, we discuss the asymptotic behaviour of the solutions for problem (1)–(3). For this purpose we define the energy by

$$E(t) = \frac{1}{2}(\|u(t)\|_{H^1}^2 + \|u_t(t)\|_{H^1}^2) + \int_0^1 F(u_x)\mathrm{d}x, \tag{42}$$

where $F(s) = \int_0^s \sigma(y)\mathrm{d}y$.

Theorem 5. Let $\lambda > 0$, $1 < p < \infty$ and assume that

(i) either $\sigma(s)s \geq 0$ or $\sigma'(s) \geq C_0$, $s \in \mathrm{R}$, where C_0 is a constant;

(ii) $E(0) = \frac{1}{2}(\|u_0\|_{H^1}^2 + \|u_1\|_{H^1}^2) + \int_0^1 F(u_{0x})\mathrm{d}x > 0$;

(iii) $D(s) \leq b\sigma(s)s$, $s \in \mathrm{R}$, where $b > 0$ is a constant.

Then for the global $W^{2,p}$ solution $u(x,t)$ of problem (1)–(3) there exist $\delta_1 > 0$ and $M > 0$ such that

$$\|u(t)\|_{H^1}^2 + \|u_t(t)\|_{H^1}^2 + 2\int_0^1 F(u_x)\mathrm{d}x \le ME(0)\mathrm{e}^{-\delta_1 t},\quad t > 0. \tag{43}$$

Proof. Let $u(x,t)$ be a global $W^{2,p}$ solution of problem (1)–(3). Taking the L^2 inner product of (1) with u_t it follows that

$$\frac{\mathrm{d}}{\mathrm{d}t}E(t) + \lambda\|u_{xt}(t)\|_2^2 = 0, \quad t > 0. \tag{44}$$

Multiplying (44) by $\mathrm{e}^{\delta t}$ gives

$$\frac{\mathrm{d}}{\mathrm{d}t}(\mathrm{e}^{\delta t}E(t)) + \lambda \mathrm{e}^{\delta t}\|u_{xt}(t)\|_2^2 = \delta \mathrm{e}^{\delta t}E(t), \quad t > 0. \tag{45}$$

Integrating (45) over $(0,t)$ we get

$$\mathrm{e}^{\delta t}E(t) + \lambda\int_0^t \mathrm{e}^{\delta\tau}\|u_{x\tau}(\tau)\|_2^2\mathrm{d}\tau = E(0) + \frac{\delta}{2}\int_0^t \mathrm{e}^{\delta\tau}(\|u_\tau(\tau)\|_2^2 + \|u_{x\tau}(\tau)\|_2^2)\mathrm{d}\tau + \delta\int_0^t \mathrm{e}^{\delta\tau}\left(\frac{1}{2}\|u(\tau)\|_2^2 + \frac{1}{2}\|u_x(\tau)\|_2^2 + \int_0^1 F(u_x)\mathrm{d}x\right)\mathrm{d}\tau,\quad t > 0. \tag{46}$$

Case 1. If $\sigma(s)s \ge 0$, $s \in \mathrm{R}$, then $F(s) \ge 0$. Thus from assumption (iii) of Theorem 5 we have $0 \le F(s) \le b\sigma(s)s$. Using this relation and (1) we obtain

$$\begin{aligned}
&\int_0^t \mathrm{e}^{\delta\tau}\left(\frac{1}{2}\|u(\tau)\|_2^2 + \frac{1}{2}\|u_x(\tau)\|_2^2 + \int_0^1 F(u_x)\mathrm{d}x\right)\mathrm{d}\tau \\
&\le b_1\int_0^t \mathrm{e}^{\delta\tau}\left(\|u(\tau)\|_2^2 + \|u_x(\tau)\|_2^2 + \int_0^1 \sigma(u_x)u_x\mathrm{d}x\right)\mathrm{d}\tau \\
&= -b_1\int_0^t \mathrm{e}^{\delta\tau}\left[(u_{\tau\tau},u) + (u_{x\tau\tau},u_x) + \frac{\lambda}{2}\frac{\mathrm{d}}{\mathrm{d}\tau}\|u_x(\tau)\|_2^2\right]\mathrm{d}\tau \\
&= -b_1\Bigg[\mathrm{e}^{\delta\tau}\left((u_t,u) + (u_{xt},u_x) + \frac{\lambda}{2}\|u_x(t)\|_2^2\right) \\
&\quad - \left((u_1,u_0) + (u_{1x},u_{0x}) + \frac{\lambda}{2}\|u_{0x}\|_2^2\right) \\
&\quad - \int_0^t \mathrm{e}^{\delta\tau}\left(\|u_\tau(\tau)\|_2^2 + \|u_{x\tau}(\tau)\|_2^2\right)\mathrm{d}\tau \\
&\quad + \delta\int_0^t \mathrm{e}^{\delta\tau}\left((u_\tau,u) + (u_{x\tau},u_x) + \frac{\lambda}{2}\|u_x(\tau)\|_2^2\right)\mathrm{d}\tau\Bigg] \\
&\le 2b_1\int_0^t \mathrm{e}^{\delta\tau}\|u_{x\tau}(\tau)\|_2^2\mathrm{d}\tau + (1+\lambda)b_1\mathrm{e}^{\delta t}E(t) \\
&\quad + (1+\lambda)b_1E(0) + (1+\lambda)b_1\delta\textstyle\int_0^t \mathrm{e}^{\delta\tau}E(\tau)\mathrm{d}\tau,\quad t > 0,
\end{aligned} \tag{47}$$

where $b_1 = \max\left\{\frac{1}{2}, b\right\}$. Substituting inequality (47) into (46) we obtain

$$\begin{aligned}
&\mathrm{e}^{\delta t}E(t) + \lambda\int_0^t \mathrm{e}^{\delta\tau}\|u_{x\tau}(t)\|_2^2\mathrm{d}\tau \le \\
&(1 + (1+\lambda)b_1\delta)E(0) + (1+2b_1)\delta\int_0^t \mathrm{e}^{\delta\tau}\|u_{x\tau}(t)\|_2^2\mathrm{d}\tau \\
&+(1+\lambda)b_1\delta \mathrm{e}^{\delta t}E(t) + (1+\lambda)b_1\delta^2\int_0^t \mathrm{e}^{\delta\tau}E(\tau)\mathrm{d}\tau,\quad t > 0.
\end{aligned} \tag{48}$$

Take δ: $0 < \delta < \min\left\{\frac{\lambda}{1+2b_1}, \frac{1}{(1+\lambda)b_1}\right\}$, we deduce from (48) that

$$\mathrm{e}^{\delta t}E(t) \le \frac{M}{2}E(0) + \theta\delta\int_0^t \mathrm{e}^{\delta\tau}E(\tau)\mathrm{d}\tau, \tag{49}$$

where $\frac{M}{2} = \frac{1+(1+\lambda)b_1\delta}{1-(1+\lambda)b_1\delta}$ and $\theta = \frac{(1+\lambda)b_1\delta}{1-(1+\lambda)b_1\delta} < 1$. Applying the Gronwall inequality to (49) we obtain the result of (43) for $\delta_1 = (1-\theta)\delta > 0$.

Case 2. If $\sigma'(s) \ge C_0$, $s \in \mathrm{R}$, let $\tilde{\sigma}(s) = \sigma(s) - k_0 s - \sigma(0)$, where $k_0 = \min\{C_0, 0\} \le 0$. Obviously $\tilde{\sigma}(0) = 0$, $\tilde{\sigma}'(s) = \sigma'(s) - k_0 \ge 0$, $\tilde{\sigma}(s)s \ge 0$, $s \in \mathrm{R}$, and if assumption (iii) of Theorem 5 holds, then a simple calculation shows that $0 \le \tilde{F}(s) = \int_0^s \tilde{\sigma}(y)\mathrm{d}y \le b\tilde{\sigma}(s)s$, $s \in \mathrm{R}$. Therefore substituting $\sigma(s) = \tilde{\sigma}(s) + k_0 s + \sigma(0)$ into (1) and repeating the proof of Case 1 implies the conclusions of Theorem 5. The theorem thus is proved.

5. Blow up of Solutions

In this section, we consider the blow up of solutions for problem (1)–(3). For this purpose, we define the energy by (42).

Theorem 6. Assume that

(i) $\sigma(s) \in C^k(\mathrm{R})$, $\sigma(s)s \le \alpha F(s) \le -\alpha\beta|s|^{m+1}$, $k \ge 2$, $s \in \mathrm{R}$, where $\alpha > 2$, $\beta > 0$ and $m > 1$ are constants;

(ii) $u_0, u_1 \in W^{k,p}[0,1] \cap W_0^{1,p}[0,1]$, $1 < p < \infty$ such that the initial energy

$$\text{(iii)}\qquad E(0) = \frac{1}{2}(\|u_0\|_{H^1}^2 + \|u_1\|_{H^1}^2) + \int_0^1 F(u_{0x})\mathrm{d}x < 0. \tag{50}$$

Then the $W^{k,p}$ solution $u(x,t)$ blows up in finite time $\tilde{T}$, that is

$$\|u(t)\|_{H^1}^2 + \|u_t(t)\|_{H^1}^2 + \lambda\int_0^t \|u_x(\tau)\|_2^2\mathrm{d}\tau \to \infty \quad \text{as } t \to \tilde{T}^-, \tag{51}$$

where $\tilde{T}$ is different for different conditions with $\lambda \geq 0$.

Proof. By multiplying (1) by u_t and integrating the new equation in the interval $(0,1)$ we obtain

$$E'(t)+\lambda \|u_{xt}(t)\|_2^2=0,\ E(t)\leq E(0)<0,\ t\geq 0. \quad (52)$$

Let

$$H(t)=\|u(t)\|_2^2+\|u_x(t)\|_2^2+\lambda\int_0^t \|u_x(\tau)\|_2^2\,\mathrm{d}\tau, \quad (53)$$

then

$$H'(t)=2(u,u_t)+2(u_x,u_{xt})+\lambda\|u_x(t)\|_2^2, \quad (54)$$

$$\begin{aligned}
H''(t)&=2\left(\|u_t(t)\|_2^2+\|u_{xt}(t)\|_2^2+\int_0^1 u_x u_{xtt}\,\mathrm{d}x\right)\\
&\quad+2\int_0^1 u(u_{xx}+u_{xxtt}+\lambda u_{xxt}-u+(\sigma(u_x))_x)\mathrm{d}x\\
&\quad+\frac{\mathrm{d}}{\mathrm{d}t}\lambda\|u_x(t)\|_2^2\\
&=2\bigg(\|u_t(t)\|_2^2+\|u_{xt}(t)\|_2^2-\|u_x(t)\|_2^2-\|u(t)\|_2^2\\
&\quad-\int_0^1 u_x\sigma(u_x)\mathrm{d}x\bigg)\\
&\geq 2\bigg(\|u_t(t)\|_2^2+\|u_{xt}(t)\|_2^2-\|u_x(t)\|_2^2-\|u(t)\|_2^2\\
&\quad-\alpha\int_0^1 F(u_x)\mathrm{d}x\bigg)\\
&\geq 2\bigg(2\|u_t(t)\|_2^2+2\|u_{xt}(t)\|_2^2-(\alpha-2)\int_0^1 F(u_x)\mathrm{d}x\\
&\quad-2E(0)\bigg)\\
&\geq 2\bigg(2\|u_t(t)\|_2^2+2\|u_{xt}(t)\|_2^2+(\alpha-2)\beta\|u_x(t)\|_{m+1}^{m+1}\\
&\quad-2E(0)\bigg),\qquad t>0, \qquad (55)
\end{aligned}$$

where the assumption (i) of Theorem 6 and the fact that

$$\begin{aligned}
\alpha\int_0^1 F(u_x)\mathrm{d}x &\leq 2E(0)-\|u_t(t)\|_2^2-\|u_x(t)\|_2^2\\
&\quad-\|u_{xt}(t)\|_2^2-\|u(t)\|_2^2+(\alpha-2)\int_0^1 F(u_x)\mathrm{d}x
\end{aligned}$$

have been used. Taking (55) and integrating this, we obtain

$$\begin{aligned}
&H'(t)\geq\\
&2(\alpha-2)\beta\int_0^t\|u_x(\tau)\|_{m+1}^{m+1}\mathrm{d}\tau-4E(0)t+H'(0), \quad (56)\\
&t>0
\end{aligned}$$

After this calculation, we could add (55) with (56). Then we get

$$\begin{aligned}
&H''(t)+H'(t)\\
&\geq 4\|u_t(t)\|_2^2+4\|u_{xt}(t)\|_2^2\\
&\quad+2(\alpha-2)\beta\left(\|u_x(t)\|_{m+1}^{m+1}+\int_0^t\|u_x(\tau)\|_{m+1}^{m+1}\mathrm{d}\tau\right)\\
&\quad-4E(0)(1+t)+H'(0)=g(t),\quad t>0. \qquad (57)
\end{aligned}$$

Take $r=\frac{m+3}{2}$, obviously $2<r<m+1$ and $r'=\frac{m+3}{m+1}$ (<2). By using the Young inequality and the Sobolev-Poincaré inequality, we get

$$\begin{aligned}
|(u,u_t)|&\leq\frac{1}{r}\|u(t)\|_r^r+\frac{1}{r'}\|u_t(t)\|_{r'}^{r'}\\
&\leq C_1[(\|u_x(t)\|_{m+1}^{m+1})^{\mu}+(\|u_t(t)\|_2^2)^{\mu}],
\end{aligned}$$

$$|(u,u_t)|^{\frac{1}{\mu}}\leq C_2[\|u_x(t)\|_{m+1}^{m+1}+\|u_t(t)\|_2^2],\quad t>0, \quad (58)$$

and similarly

$$|(u_x,u_{xt})|^{\frac{1}{\mu}}\leq C_3[\|u_x(t)\|_{m+1}^{m+1}+\|u_{xt}(t)\|_2^2],\ t>0, \quad (59)$$

where in this inequality and in the sequel C_i $(i=1,2,\ldots)$ denote positive constants independent of t, $\mu=\frac{m+3}{2(m+1)}$ (<1). By the Sobolev-Poincaré inequality and the Hölder inequality

$$\|u_x(t)\|_{m+1}^{m+1}\geq\left(\|u(t)\|_2^2\right)^{\frac{m+1}{2}},\quad t>0, \qquad (60)$$

$$\|u_x(t)\|_{m+1}^{m+1}\geq\left(\|u_x(t)\|_2^2\right)^{\frac{m+1}{2}},\quad t>0, \qquad (61)$$

$$\int_0^t\|u_x(\tau)\|_{m+1}^{m+1}\mathrm{d}\tau\geq t^{\frac{1-m}{2}}\left(\int_0^t\|u_x(\tau)\|_2^2\mathrm{d}\tau\right)^{\frac{m+1}{2}},$$
$$t>0. \qquad (62)$$

(1) If $\lambda>0$, by using the inequalities (58)–(62), we obtain

$$\begin{aligned}
g(t)\geq C_4\bigg(&4\|u_x(t)\|_{m+1}^{m+1}+\|u_t(t)\|_2^2+\|u_{xt}(t)\|_2^2\\
&+\int_0^t\|u_x(\tau)\|_{m+1}^{m+1}\mathrm{d}\tau\bigg)-4E(0)t+H'(0)
\end{aligned}$$

$$\geq C_5\left(|(u,u_t)|^{\frac{1}{\mu}}+|(u_x,u_{xt})|^{\frac{1}{\mu}}+\left(\|u(t)\|_2^2\right)^{\frac{m+1}{2}}\right.$$
$$\left.+\left(\|u_x(t)\|_2^2\right)^{\frac{m+1}{2}}+t^{\frac{1-m}{2}}\left(\int_0^t\|u_x(\tau)\|_2^2\,\mathrm{d}\tau\right)^{\frac{m+1}{2}}\right)$$
$$-4E(0)t+H'(0)$$
$$\geq C_6t^{\frac{1-m}{2}}\left(|(u,u_t)|^{\gamma}+|(u_x,u_{xt})|^{\gamma}+\left(\|u(t)\|_2^2\right)^{\gamma}\right.$$
$$\left.+\left(\|u_x(t)\|_2^2\right)^{\gamma}+\left(\int_0^t\|u_x(\tau)\|_2^2\,\mathrm{d}\tau\right)^{\gamma}\right)$$
$$-4E(0)t+H'(0)-C_6t^{\frac{1-m}{2}},\quad t\geq 1, \tag{63}$$

where in this inequality and in the sequel $\gamma=\frac{1}{\mu}>1$. Since $-4E(0)t+H'(0)-C_6t^{\frac{1-m}{2}}\to\infty$ as $t\to\infty$, there must be a $t_1\geq 1$ such that

$$-4E(0)t+H'(0)-C_6t^{\frac{1-m}{2}}\geq 0 \text{ as } t\geq t_1. \tag{64}$$

Let

$$y(t)=H'(t)+H(t), \tag{65}$$

then from (56) and (53) we obtain $y(t)>0$ as $t\geq t_1$. By using the inequality

$$(a_1+\ldots+a_l)^n\leq 2^{(n-1)(l-1)}(a_1^n+\ldots+a_l^n),$$

where $a_i\geq 0$ $(i=1,\ldots,l)$ and $n>1$ are real numbers, the fact (64) and using (63), we get

$$g(t)\geq C_6t^{\frac{1-m}{2}}y^{\gamma}(t),\quad t\geq t_1. \tag{66}$$

So combining (57) with (66) gives

$$y'(t)\geq C_6t^{\frac{1-m}{2}}y^{\gamma}(t),\quad t\geq t_1. \tag{67}$$

Therefore, there exists a positive constant

$$\tilde{T}\leq\begin{cases}\left[t_1^{\frac{3-m}{2}}+\dfrac{3-m}{2C_6(\gamma-1)y^{\gamma-1}(t_1)}\right]^{\frac{2}{3-m}}, & m\neq 3,\\ t_1\cdot\exp\dfrac{1}{C_6(\gamma-1)y^{\gamma-1}(t_1)}, & m=3,\end{cases} \tag{68}$$

such that

$$y(t)\to\infty \text{ as } t\to\tilde{T}^-. \tag{69}$$

By using (53), (54), and (69), we find

$$2\|u(t)\|_2^2+\|u_t(t)\|_2^2+(\lambda+2)\|u_x(t)\|_2^2+\|u_{xt}(t)\|_2^2$$
$$+\lambda\int_0^t\|u_x(\tau)\|_2^2\,\mathrm{d}\tau\geq H'(t)+H(t)\to\infty \tag{70}$$
as $t\to\tilde{T}^-$.

So (70) implies (51).

(2) If $\lambda=0$ by using the inequalities (58) – (61), we obtain

$$g(t)\geq C_7\left(4\|u_x(t)\|_{m+1}^{m+1}+\|u_t(t)\|_2^2+\|u_{xt}(t)\|_2^2+1\right)$$
$$-4E(0)t+H'(0)$$
$$\geq C_8\left[|(u,u_t)|^{\gamma}+|(u_x,u_{xt})|^{\gamma}+\left(\|u(t)\|_2^2\right)^{\gamma}\right.$$
$$\left.+\left(\|u_x(t)\|_2^2\right)^{\gamma}\right]-4E(0)t+H'(0),$$
$$t>0. \tag{71}$$

By the same method as used in deriving (67), there must be a $t_2>0$ such that $-4E(0)t+H'(0)>0$ and $y(t)=H'(t)+H(t)>0$ as $t\geq t_2$. So combining (57) with (71) yields

$$y'(t)\geq C_8y^{\gamma}(t),\quad t\geq t_2. \tag{72}$$

Equation (72) implies that there exists a positive constant $\tilde{T}=t_2+[C_9(\gamma-1)y^{\gamma-1}(t_2)]^{-1}$ such that $y(t)\to\infty$ as $t\to\tilde{T}^-$. Since $y(t)\leq 2\|u(t)\|_2^2+\|u_t(t)\|_2^2+2\|u_x(t)\|_2^2+\|u_{xt}(t)\|_2^2$, (51) is satisfied. This completes the proof.

Theorem 7. Suppose that the conditions (i) and (ii) of Theorem 6 hold and one of the following conditions are valid:

(i) $E(0)=0$, and, if $\lambda>0$, $H'(0)\geq C_{10}t^{\frac{1-m}{2}}$ for $t\geq 1$, and, if $\lambda=0$, $H'(0)>0$ for $t>0$.

(ii) $E(0)>0$, and, if $\lambda>0$, $H'(0)\geq 4E(0)(1+t)+C_{10}t^{\frac{1-m}{2}}$ for some $t_3\geq 1$, and, if $\lambda=0$, $H'(0)>4E(0)(1+t)$ for some $t_4>0$.

Then the $W^{k,p}$ solution $u(x,t)$ blows up in finite time $\tilde{T}$.

Proof. At first we assume that condition (i) holds.

(1) If $\lambda>0$ then using inequalities (58) – (62), we obtain

$$g(t)\geq C_9t^{\frac{1-m}{2}}\left(|(u,u_t)|^{\gamma}+|(u_x,u_{xt})|^{\gamma}+\left(\|u(t)\|_2^2\right)^{\gamma}\right.$$
$$\left.+\left(\|u_x(t)\|_2^2\right)^{\gamma}+\left(\int_0^t\|u_x(\tau)\|_2^2\,\mathrm{d}\tau\right)^{\gamma}\right) \tag{73}$$
$$-4E(0)(1+t)+H'(0)-C_9t^{\frac{1-m}{2}},\quad t\geq 1.$$

From condition (i) we have

$$-4E(0)(1+t)+H'(0)-C_9t^{\frac{1-m}{2}}=H'(0)-C_9t^{\frac{1-m}{2}}$$
$$\geq 0 \text{ as } t\geq 1. \tag{74}$$

Thus, we obtain $y(t) > 0$ as $t \geq 1$. By the similar method as used in deriving (67), we find

$$y'(t) \geq C_{10} t^{\frac{1-m}{2}} y^{\gamma}(t), \quad t \geq 1. \tag{75}$$

Therefore there exists a positive constant

$$\tilde{T} \leq \begin{cases} \left[1 + \dfrac{3-m}{2C_9(\gamma-1)y^{\gamma-1}(1)}\right]^{\frac{2}{3-m}}, & m \neq 3, \\ \exp \dfrac{1}{C_9(\gamma-1)y^{\gamma-1}(1)}, & m = 3, \end{cases} \tag{76}$$

such that $y(t) \to \infty$ as $t \to \tilde{T}^-$.

(2) If $\lambda = 0$ then using inequalities (58)–(61), we obtain

$$g(t) \geq C_{10}\Big[|(u,u_t)|^{\gamma} + |(u_x,u_{xt})|^{\gamma} + \left(\|u(t)\|_2^2\right)^{\gamma} + \left(\|u_x(t)\|_2^2\right)^{\gamma}\Big] - 4E(0)(1+t) + H'(0), \quad t > 0. \tag{77}$$

From condition (i), we have

$$-4E(0)(1+t) + H'(0) = H'(0) > 0 \text{ as } t > 0. \tag{78}$$

Thus we obtain $y(t) > 0$ as $t > 0$. As a result, we get

$$y'(t) \geq C_{11} y^{\gamma}(t), \quad t > 0. \tag{79}$$

Equation (79) implies that there exists a positive constant $\tilde{T} \leq [C_{11}(\gamma-1)y^{\gamma-1}(0)]^{-1}$ such that $y(t) \to \infty$ as $t \to \tilde{T}^-$.

Second, we assume that condition (ii) holds.

(1) If $\lambda > 0$ from condition (ii) we have

$$-4E(0)(1+t) + H'(0) - C_{10} t^{\frac{1-m}{2}} \geq 0 \text{ as } t \geq t_3. \tag{80}$$

By use of inequalities (58)–(62) and (80), we obtain $y(t) > 0$ as $t > t_3$. Thus, we find

$$y'(t) \geq C_{10} t^{\frac{1-m}{2}} y^{\gamma}(t), \quad t \geq t_3. \tag{81}$$

Therefore, there exists a positive constant

$$\tilde{T} \leq \begin{cases} \left[t_3^{\frac{3-m}{2}} + \dfrac{3-m}{2C_{10}(\gamma-1)y^{\gamma-1}(t_3)}\right]^{\frac{2}{3-m}}, & m \neq 3, \\ t_3 \cdot \exp \dfrac{1}{C_{10}(\gamma-1)y^{\gamma-1}(t_3)}, & m = 3, \end{cases} \tag{82}$$

such that $y(t) \to \infty$ as $t \to \tilde{T}^-$.

(2) If $\lambda = 0$ from condition (ii) we have

$$-4E(0)(1+t) + H'(0) > 0 \text{ as } t \geq t_4. \tag{83}$$

By use of inequalities (58)–(61) and (83), we obtain $y(t) > 0$ as $t \geq t_4$. It follows that

$$y'(t) \geq C_{11} y^{\gamma}(t), \quad t \geq t_4. \tag{84}$$

Equation (84) implies that there exists a positive constant $\tilde{T} \leq t_4 + [C_{11}(\gamma-1)y^{\gamma-1}(t_4)]^{-1}$ such that $y(t) \to \infty$ as $t \to \tilde{T}^-$. The theorem is proved.

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